Taylor Polynomials continued.

Given a smooth (all derivatives exist (and are continuous)) function $f: \mathbb{R} \to \mathbb{R}$, the Taylor polynomial $P_n(x)$ of degree n at $x=a$ is the degree n polynomial that fits f best near $x=a$, and

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{j=0}^{n} \frac{f^{(j)}(a)}{j!}(x-a)^j .$$

Examples (If not mentioned, assume $a=0$ is the point in question)

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!}$$

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!}x^n$$

Interesting tricks

① $f(x) = \frac{1}{2}(x-1)^2 + x^7 \leftarrow$ $P_{23}(x) =$ degree 23 Taylor polynomial at $x = 17.4$.

$$P_{23}(x) = \frac{1}{2}(x-1)^2 + x^7$$

② $f(x) = x^5 e^{x^7}$ $P_{25}(x) \leftarrow$ degree 25 Taylor poly at $x = 0$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{6} + \cdots$$

$$P_{25}(x) = x^5\left(1 + (x^7) + \frac{(x^7)^2}{2} + \frac{(x^7)^3}{6} + \cdots\right)$$

$$= x^5\left(1 + x^7 + \frac{x^{14}}{2} + \frac{x^{21}}{6} + \frac{x^{28}}{4!}\cdots\right)$$

$$= \boxed{x^5 + x^{12} + \frac{x^{19}}{2}}$$

Taylor-Lagrange Remainder formula

Idea

$$f(x) \approx P_n(x) = f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

exact   approximation

What's the error?

If $f : \mathbb{R} \to \mathbb{R}$ is smooth near $x = a$,

then

$$f(x) = f(a) + f'(a)(x-a) + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

where c is between a & x

(a specific # somewhere between a & x).

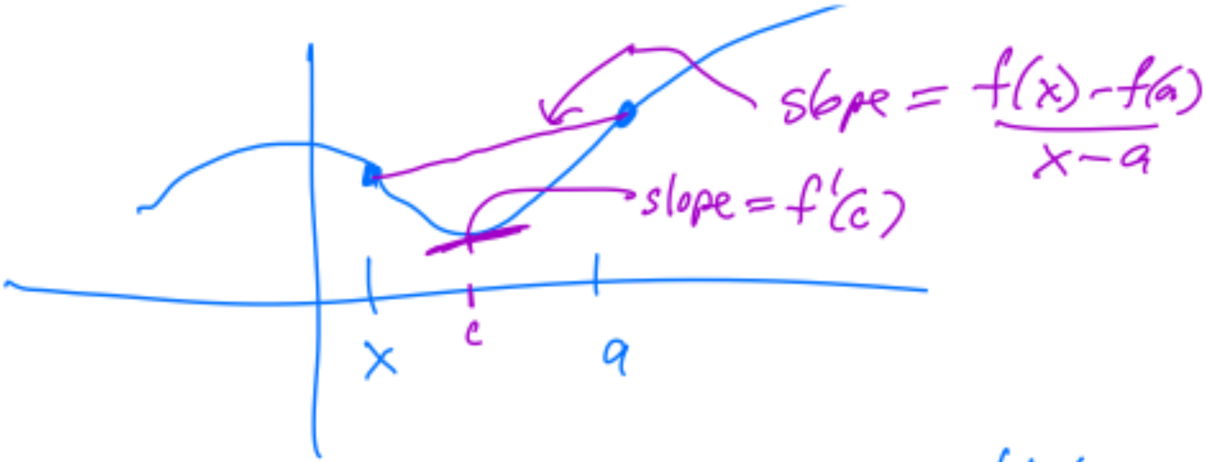
When $n=0$, this says

$$f(x) = f(a) + f'(c)(x-a) \text{ for some } c$$

between a & x.

$$f(x) - f(a) = f'(c)(x-a)$$

$$\Rightarrow \quad f'(c) = \frac{f(x) - f(a)}{x-a} \qquad \text{Mean Value Theorem.}$$



The Taylor-Lagrange Remainder is a higher degree version of the mean value theorem.

The theorem is useful to us, because it helps us estimate errors.

<u>Example:</u> Estimate $\cos(0.03)$ with the $4^{\underline{th}}$ degree Taylor polynomial. Estimate the error in this approximation.

Taylor for $\cos(x)$:

$$P_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} = 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

approx:

$$P_4(0.03) = 1 - \frac{(.03)^2}{2} + \frac{(.03)^4}{24}$$

$x = 0.03$
$a = 0$

$$= 1 - \frac{.0009}{2} + \frac{.000000\cancel{81}^{27}}{\cancel{24}\ 8}$$

$= 1 - .00045$ $\quad \frac{27}{8} = 3\frac{3}{8}$

$+ .0000003375$ $\quad 3.375$

$= .99955 + .0000003375$

$P_4(0.03) = .9995503375 = \bar{x}$

$x = \cos(.03)$

Error term

$$f(x) - P_4(x) = \frac{f^{(5)}(c)}{5!}(.03)^5$$

where $0 < c < 0.03$

$f(x) = \cos(x)$
$f'(x) = -\sin(x)$
$f''(x) = -\cos(x)$
$f'''(x) = \sin(x)$
$f^{(4)}(x) = \cos(x)$
$f^{(5)}(x) = -\sin(x)$

$$|e_x| = |f(x) - P_4(x)| = \left|\frac{f^{(5)}(c)}{120}(.03)^5\right|$$

$$= \left|\frac{-\sin(c)\,(0.03)^5}{120}\right| \leq \frac{1\,(0.03)^5}{120}$$

$$= \frac{.0000000\cancel{243}\;081}{\cancel{120}\;40}$$

$\frac{81}{40} = 2\frac{1}{40}$

$$\boxed{|e_x| \leq .00000000002025}$$

2.025

$\therefore$ Our estimate for $\cos(0.03)$ must be accurate at least to 9 dec. places

$$\cos(0.03) = \underbrace{.999550033}75$$

<u>Application</u>: If f is a smooth function, and we estimate x by $\bar{x}$, we can determine how accurate the approximation $f(\bar{x})$ is to $f(x)$.

$$e_{f(x)} = ?$$

$$e_{f(x)} = f(x) - f(\bar{x})$$

(Taylor-h formula)

$$\Rightarrow e_{f(x)} = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{f''(c)(x-\bar{x})^2}{2} - f(\bar{x})$$

Taylor @ $x=\bar{x}$

$$f(x) = f(\bar{x}) + f'(\bar{x})(x-\bar{x}) + \frac{f''(c)(x-\bar{x})^2}{2}$$

$$\Rightarrow e_{f(x)} = f'(\bar{x})(x-\bar{x}) + \frac{f''(c)}{2}(x-\bar{x})^2$$

$$\Rightarrow \boxed{e_{f(x)} = f'(\bar{x})\,e_x + \frac{f''(c)}{2}e_x^2}$$

for some c between x & $\bar{x}$.

$$\Rightarrow \quad |e_{f(x)}| = \left| f'(\bar{x}) e_x + \frac{f''(c)}{2} e_x^2 \right|$$
$$\leq |f'(\bar{x}) e_x| + \left| \frac{f''(c)}{2} e_x^2 \right|$$
$$= |f'(\bar{x})|\,|e_x| + \frac{|f''(c)|}{2} |e_x|^2$$
$$\Rightarrow \boxed{|e_{f(x)}| \leq |f'(\bar{x})|\,|e_x| + \frac{|f''(c)|}{2} |e_x|^2}$$

typically smaller than $|e_x|$.

$$|e_{f(x)}| \approx |f'(\bar{x})|\,|e_x|$$

If we have a bound $|f''(c)| \leq K$, then we can definitely say

$$\boxed{|e_{f(x)}| \leq |f'(\bar{x})|\,|e_x| + \frac{K}{2} |e_x|^2.}$$

Relative Error

$e_{f(x)} = f(\bar{x}) \varepsilon_{f(x)}$
$e_x = \bar{x} \varepsilon_x$

$$e_{f(x)} = f'(\bar{x}) e_x + \frac{f''(c)}{2} e_x^2$$
$$\Rightarrow f(\bar{x}) \varepsilon_{f(x)} = f'(\bar{x}) \bar{x} \varepsilon_x + \frac{f''(c)}{2} \bar{x}^2 \varepsilon_x^2$$

assume $f(\bar{x}) \neq 0$

$$\Rightarrow \mathcal{E}_{f(x)} = \frac{f'(\bar{x})}{f(\bar{x})}\bar{x}\,\mathcal{E}_x + \frac{f''(c)}{2f(\bar{x})}\bar{x}^2\mathcal{E}_x^2$$

$$|\mathcal{E}_{f(x)}| = \left|\frac{f'(\bar{x})}{f(\bar{x})}\bar{x}\,\mathcal{E}_x + \frac{f''(c)}{2f(\bar{x})}\bar{x}^2\mathcal{E}_x^2\right|$$

$$\leq \left|\frac{f'(\bar{x})}{f(\bar{x})}\bar{x}\,\mathcal{E}_x\right| + \left|\frac{f''(c)}{2f(\bar{x})}\bar{x}^2\mathcal{E}_x^2\right|$$

$$= \left|\frac{f'(\bar{x})}{f(\bar{x})}\right||\bar{x}||\mathcal{E}_x| + \frac{|f''(c)|}{2|f(\bar{x})|}|\bar{x}|^2|\mathcal{E}_x|^2$$

$$\left(\log(f(\bar{x}))\right)' = \frac{f'(\bar{x})}{f(\bar{x})}$$

$$\Rightarrow |\mathcal{E}_{f(x)}| \leq \left(\left|(\log f)'(\bar{x})\right||\bar{x}|\right)|\mathcal{E}_x| + \left(\frac{|f''(c)|\,|\bar{x}|^2}{2|f(\bar{x})|}\right)|\mathcal{E}_x|^2$$

$$|\mathcal{E}_x|^2 << |\mathcal{E}_x|$$

$$\Rightarrow |\mathcal{E}_{f(x)}| \lesssim \left|(\log f)'(\bar{x})\right||\bar{x}|\,|\mathcal{E}_x|$$

good bound!

$(\log f)'(\bar{x}) = \frac{f'(\bar{x})}{f(\bar{x})}$

$(\ln f)'(\bar{x}) = [\ln(f(\bar{x}))]'$

This gives a general formula for the bound on $\mathcal{E}_{f(x)}$.

In text: the general formula for the error in $f(x,y,\dots)$ is derived.

Example: If $x = 2.581 \pm .002$, give the value of e^x along with the error bounds.

Solution: $\left.\begin{aligned}\bar{x} &= 2.581 \\ |e_x| &\le .002\end{aligned}\right]$ given

from our error bounds,

$|e_{f(x)}| \le ?$

If we have a bound $|f''(c)| \le K$, then we can definitely say

$$\boxed{|e_{f(x)}| \le |f'(\bar{x})|\,|e_x| + \frac{K}{2}|e_x|^2.}$$

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$$|e_{f(x)}| \le |e^{\bar{x}}|\,.002 + \frac{|e^c|}{2}(.002)^2$$

$x \approx \bar{x} = 2.581 \pm .002 \le 2.583$

e^x increasing $\Longrightarrow e^{\bar{x}} \le e^{2.583},\ e^c \le e^{2.583}$

$$e^{2.583} = 13.2368\ldots$$

$$|e_{f(x)}| \le (13.2368)(.002) + \frac{13.2368}{2}(.002)^2$$

$$.000002$$

$$|e_{f(x)}| \le .0265 + .0000265$$

$$\boxed{|e_{f(x)}| \le .0265265}$$

$$e^{2.581} = 13.2103$$

Thus $\boxed{e^x = 13.2103 \pm .0266}$

In this example, we can more easily find the error difference

$$2.579 \le x \le 2.583$$

$$\Rightarrow \quad e^{2.579} \le e^x \le e^{2.583}$$

(e^x is increasing $\Rightarrow$ it preserves inequalities)

$$\boxed{13.1839 \le e^x \le 13.2368}$$